INVERSE TRIGONOMETRIC FUNCTIONS

Before starting about inverse trigonometric function let's briefly discuss about what is inverse of any function. Corresponding to every Bijection (one-one and onto), $f: A \to B,$ there exists another bijection. It means if we interchange the domain and range of f and let the new fuction denoted by g which will be given by $g: B \to A$ defined by $g(y) = x$ if and only if $f(x) = y$. So $g: B \to A$ is called as the inverse of $f: A \to B$ denoted by f^{-1}.

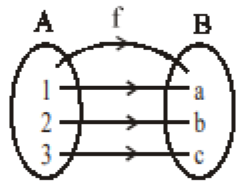

Fig 2.12

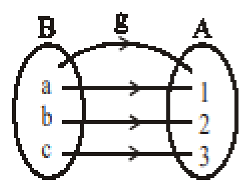

Fig 2.13

A function $f: A \to B$ is invertible it means f^{-1}exists if it is one-one and onto. Consider the case of trigonometric functions. In case of sine function: $sin: R \to [-1,1]$. But $\sin 0 = \sin \pi \ = 0$ where $0 \neq \pi$. Thus sine function is not bijective in the domain R. However we see that for $y \ \in [-1,1]$ there exists a unique number x in each of the intervals$\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right], \left[--\frac{\pi}{2}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, ..., such that $\sin x = y$. So if $\sin x = \theta \leftrightarrow x = \sin^{-1}\theta$, and read as "sin inverse of x". The function $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \sec^{-1} x, cosec^{-1}x, \cot^{-1} x$ are called inverse trigonometric function.

Example: We know the $\sin 30^{\circ} = \frac{1}{2}$, when the angle is expressed in degrees and $\sin\frac{\pi}{6} = \frac{1}{2}$, when the angle expressed in radians. It means that, sine of the angle $\pi/6$ radian is $1/2$. The converse statement is, the angle whose sine is ½ is $\pi/6$ radian. Symbolically, it is written as $sin^{-1}\frac{1}{2}$ (whose value is $\pi/6$). The function $\sin^{-1}\theta$ is an number, where as $sin\theta$ is a real number.

Function	Domain(D)	Range(R)
$\sin^{-1} x$	$-1 \leq x \leq 1 \ or \ [-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$-1 \leq x \leq 1 \ or \ [-1,1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
$\mathrm{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Properties of Inverse Trigonometric Functions

1. Self adjusting property:

(i) $\boldsymbol{\sin^{-1}(sin\theta) = \theta}$

(ii) $\boldsymbol{cos^{-1}(cos\theta) = \theta}$

(iii) $\boldsymbol{tan^{-1}(tan\theta) = \theta}$

Proof: (i) Let $sin\theta = x,\ then\ \theta = sin^{-1}x$.

$\therefore \sin^{-1}(sin\theta) = \sin^{-1} x = \theta$ (Proved)

Similarly, proofs of (ii) & (iii) can be completed..

2. Reciprocal Property:

i. $\mathbf{cosec^{-1}\frac{1}{x} = \sin^{-1} x}$

ii. $\mathbf{sec^{-1}\frac{1}{x} = \cos^{-1} x}$

iii. $\mathbf{cot^{-1}\frac{1}{x} = \tan^{-1} x}$

Proof :

(i) Let $\sin^{-1} x = \theta, \Rightarrow x = sin\theta$

Then, $cosec\theta = \frac{1}{\sin\theta} = \frac{1}{x}, \Rightarrow \theta = cosec^{-1}\frac{1}{x}$

Hence, $= \sin^{-1} x$ and $\theta = \text{cosec}^{-1}\frac{1}{x}$,

Therefore, $\sin^{-1} x = cosec^{-1}\frac{1}{x}$

$\therefore \sin^{-1} x = \text{cosec}^{-1}\frac{1}{x}$ (Proved)

Similarly (ii) and (iii) can be proved.

3. Conversion property:

(i) $\mathbf{\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}}$

(ii) $\mathbf{\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}}$

Proof:

(i) Let $\theta = \sin^{-1} x$ so that $\sin x = \theta$

Now $cos\theta = \sqrt{1 - sin^2\theta} = \sqrt{1-x^2}$

i.e. $\theta = cos^{-1}\sqrt{1-x^2}$

$\therefore$ $\tan\theta = \frac{sin\theta}{cos\theta} = \frac{x}{\sqrt{1-x^2}}$ Or, $\theta = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

Thus, $\theta = \sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$ (Proved)

Similalr (ii) and (iii) can also be proved.

4. Theorem – 1:

(i) $\mathbf{\sin^{-1} x + \cos^{-1} x = \pi/2}$

(ii) $\mathbf{\tan^{-1} x + \cot^{-1} x = \pi/2}$

(iii) $\mathbf{\sec^{-1} x + cosec^{-1} x = \pi/2}$

Proof:

(i) Let $\sin^{-1} x = \theta$,

$\Rightarrow x = sin\theta = \cos(\pi/2 - \theta)$

$\Rightarrow \cos^{-1} x = {}^{\pi}/_{2} - \theta = {}^{\pi}/_{2} - \sin^{-1} x$

$\Longrightarrow \sin^{-1} x + \cos^{-1} x = \pi/2$ (Proved)

Similarly (ii) and (iii) can also be proved.

1. **Theorem – 2: If $xy < 1$, then $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$**

Proof:
Let $\tan^{-1} x = \theta_1$ *and* $\tan^{-1} y = \theta_2$ then
$x = \tan\theta_1$ *and* $y = \tan\theta_2$
Now, $\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$
$\Rightarrow \tan(\theta_1 + \theta_2) = \frac{x+y}{1-xy}$
$\Rightarrow (\theta_1 + \theta_2) = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$
$\Rightarrow \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ (Proved)

2. **Theorem – 3: $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$**

Proof:
Let $\tan^{-1} x = \theta_1$ *and* $\tan^{-1} y = \theta_2$
then $x = \tan\theta_1$ *and* $y = \tan\theta_2$
Now, $\tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$
$\Rightarrow \tan(\theta_1 - \theta_2) = \frac{x-y}{1+xy}$
$\Rightarrow (\theta_1 - \theta_2) = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$
$\Rightarrow \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ (Proved)

Note:

1. $\mathbf{\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)}$
2. $\mathbf{2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}\ \textit{if}\ |x| < 1}$
   $\mathbf{= \sin^{-1}\frac{2x}{1+x^2}\ \textit{if}|x| \le 1}$
   $\mathbf{= \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\ \textit{if}\ |x| \ge 0}$

3. **Theorem – 4:**

(i) $\mathbf{2\sin^{-1} x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]}$
(ii) $\mathbf{2\cos^{-1} x = \cos^{-1}\left[2x^2 - 1\right]}$

Proof :
(i) $\sin^{-1} x = \theta$, then $x = \sin\theta$
$\therefore sin2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1 - sin^2\theta} = 2x\sqrt{1-x^2}$
$\Rightarrow 2\theta = \sin^{-1} 2x\sqrt{1-x^2}$
$\Rightarrow 2\sin^{-1} x = \sin^{-1} 2x\sqrt{1-x^2}$ (Proved)
(ii). Let $\cos^{-1} x = \theta$ then $x = \cos\theta$
$\therefore cos2\theta = 2cos^2\theta - 1$
$\Rightarrow cos2\theta = 2x^2 - 1$

$\Rightarrow 2\theta = \cos^{-1}(2x^2 - 1)$
$\Rightarrow 2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$ (Proved)

4. Theorem-5:

(i) $\mathbf{3\sin^{-1} x = \sin^{-1}(3x - 4x^3)}$
(ii) $\mathbf{3\cos^{-1} x = \cos^{-1}(3x - 4x^3)}$
(iii) $\mathbf{3\tan^{-1} x = \tan^{-1}\frac{3x-x^3}{1-3x^2}}$

Proof:
(i) Let $sin^{-1}x = \theta$, $\Longrightarrow x = \sin\theta$
We know that,
$\sin 3\theta = 3sin\theta - 4sin^3\theta$
$\Longrightarrow \sin 3\theta = 3x - 4x^3$
$\Longrightarrow 3\theta = sin^{-1}(3x - 4x^3)$
$\Longrightarrow 3sin^{-1}x = sin^{-1}(3x - 4x^3)$
Similarly, (ii) and (iii) can also be proved.

5. Theorem-6:

(i) $\mathbf{\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)}$
(ii) $\mathbf{\cos^{-1} x + \cos^{-1} y = \cos^{-1} xy - \sqrt{1-x^2}\sqrt{1-y^2}}$
(iii) $\mathbf{\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)}$
(iv) $\mathbf{\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)}$

Proof:
(i) Let $\sin^{-1} x = \theta_1$, and $\sin^{-1} y = \theta_2$
Then, $x = \sin\theta_1$ and $y = \sin\theta_2$
$\therefore \sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1 sin\theta_2$
$= sin\theta_1\sqrt{1 - sin^2\theta_2} + \sqrt{1 - sin^2}\theta_1 \sin\theta_2$
$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$
$\Rightarrow \theta_1 + \theta_2 = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$
$\Rightarrow \sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

Similarly, others can also be proved.

6. Theorem – 7

(i) $\boldsymbol{sin^{-1}(-x) = -sin^{-1}x}$
(ii) $\boldsymbol{cos^{-1}(-x) = \pi - cos^{-1}x}$
(iii) $\boldsymbol{tan^{-1}(-x) = -tan^{-1}x}$

Proof:
(i) Let $-x = \sin\theta$, $\Longrightarrow \theta = sin^{-1}(-x)$ (1)
Since, $-x = \sin\theta$,
$\Longrightarrow x = -\sin\theta = \sin(-\theta)$
$\Longrightarrow -\theta = sin^{-1}x$

$\Longrightarrow \theta = -sin^{-1}x$ (2)

From eqn. (1) and (2), $sin^{-1}(-x) = -sin^{-1}x$ (Proved)

(ii) Let $-x = \cos\theta$, $\Longrightarrow \theta = cos^{-1}(-x)$ (3)

Since, $-x = \cos\theta$,

$\Longrightarrow x = -\cos\theta = \cos(\pi - \theta)$

$\Longrightarrow \pi - \theta = cos^{-1}x$

$\Longrightarrow \theta = \pi - cos^{-1}x$ (4)

From eqn. (3) and (4), $cos^{-1}(-x) = \pi - cos^{-1}x$

(iii) Let $-x = \tan\theta$, $\Longrightarrow \theta = tan^{-1}(-x)$ (5)

Since, $-x = \tan\theta$,

$\Longrightarrow x = -\tan\theta = \tan(-\theta)$

$\Longrightarrow -\theta = tan^{-1}x$

$\Longrightarrow \theta = -tan^{-1}x$ (6)

From eqn. (5) and (6), $tan^{-1}(-x) = -tan^{-1}x$

Some Solved Problems:

Q– 1: Find the value of $\cos\tan^{-1}\cot\cos^{-1}\sqrt{3}/2$

Sol:

$\cos\tan^{-1}\cot\cos^{-1}\sqrt{3}/2$

$= \cos\tan^{-1}\cot\cos^{-1}\cos{}^{\pi}/_{6}$

$= \cos\tan^{-1}\cot{}^{\pi}/_{6}$

$= \cos\tan^{-1}\tan\left({}^{\pi}/_{2} - {}^{\pi}/_{6}\right)$

$= \cos\left({}^{\pi}/_{2} - {}^{\pi}/_{6}\right)$

$= \sin\pi/6 = 1/2$

Q-2: Prove that $tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{3} = \frac{\pi}{4}$

Proof:

LHS= $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$

$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)$ $\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= \tan^{-1}\left(\frac{5/6}{1-1/6}\right) = \tan^{-1}\left(\frac{5/6}{5/6}\right) = \tan^{-1}(1) = \frac{\pi}{4}$ = RHS

Q-3: Prove that $sin^{-1}\frac{4}{5} + sin^{-1}\frac{5}{13} = cos^{-1}\frac{16}{65}$

Proof:

LHS $= sin^{-1}\frac{4}{5} + sin^{-1}\frac{5}{13}$

$= sin^{-1}\left[\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right]$ $\left[\begin{array}{c}\because \sin^{-1}x + sin^{-1}y \\ = sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)\end{array}\right]$

$= sin^{-1}\left[\frac{4}{5}\sqrt{1-\frac{25}{169}}+\frac{5}{13}\sqrt{1-\frac{16}{25}}\right]$

$= sin^{-1}\left[\frac{4}{5}\times\frac{12}{13}+\frac{5}{13}\times\frac{3}{5}\right]$

$= sin^{-1}\left(\frac{63}{65}\right) = cos^{-1}\sqrt{1-\left(\frac{63}{65}\right)^2} = cos^{-1}\frac{16}{65}$ = RHS

Q-4: Prove that $2tan^{-1}\frac{1}{3} = tan^{-1}\frac{3}{4}$

Proof:

We know that $2\ tan^{-1}x = tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$\therefore\ 2tan^{-1}\frac{1}{3} = tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right) = tan^{-1}\left(\frac{2/3}{8/9}\right) = tan^{-1}\frac{3}{4}$ =RHS

Q-5: Show that $sin^{-1}\frac{12}{13}+cos^{-1}\frac{4}{5}+tan^{-1}\frac{63}{16} = \pi$

Proof:

LHS= $sin^{-1}\frac{12}{13}+cos^{-1}\frac{4}{5}+tan^{-1}\frac{63}{16}$

$= tan^{-1}\left(\frac{12/13}{\sqrt{1-(12/13)^2}}\right)+tan^{-1}\left(\frac{\sqrt{1-(4/5)^2}}{4/5}\right)+tan^{-1}\frac{63}{16}$ $\left[\because\ sin^{-1}x = tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right),\ cos^{-1}x = tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right]$

$= tan^{-1}\left(\frac{12}{5}\right)+tan^{-1}\left(\frac{3}{4}\right)+tan^{-1}\frac{63}{16}$

$= tan^{-1}\left(\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right)+tan^{-1}\frac{63}{16}$ $\left[\because tan^{-1}x+tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= tan^{-1}\left(-\frac{63}{16}\right)+tan^{-1}\frac{63}{16}$

$= tan^{-1}0 = \pi$ = RHS

Q – 6: Prove that $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Proof:

L.H.S = $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{2\times\frac{1}{2}}{1-\left({}^1/_2\right)^2}+\tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{1}{\left(\frac{3}{4}\right)}+\tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\left(\frac{4}{3}\right)\left(\frac{1}{7}\right)}$

$= \tan^{-1} \frac{\left({}^{31}/_{21}\right)}{\left({}^{17}/_{21}\right)}$

$= \tan^{-1} \frac{31}{17}$ = R.H.S

Q – 7: Prove that $\cot^{-1} 9 + \operatorname{cosec}^{-1} \sqrt{41}/4 = \pi/4$

Proof:

L.H.S.= $\cot^{-1} 9 + \operatorname{cosec}^{-1} \frac{\sqrt{41}}{4}$

$= \tan^{-1} \frac{1}{9} + \tan^{-1} \frac{4}{5} \quad (\because \operatorname{cosec}^{-1} \frac{\sqrt{41}}{4} = \tan^{-1} \frac{4}{5})$

$= \tan^{-1} \frac{\frac{1}{9} + \frac{4}{5}}{1 - \left(\frac{1}{9}\right)\left(\frac{4}{5}\right)}$

$= \tan^{-1} \frac{41/45}{41/45} = \tan^{-1} 1 = \pi/4$= R.H.S

Q – 8: If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi,$ then prove that $x^2 + y^2 + z^2 + 2xyz = 1$

Proof:

Given $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$

$\Rightarrow \cos^{-1} x + \cos^{-1} y = \pi - \cos^{-1} z$

$\Rightarrow \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \pi - \cos^{-1} z$

$\Rightarrow \left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \cos(\pi - \cos^{-1} z)$

$\Rightarrow \left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = -\cos(\cos^{-1} z)$

$\Rightarrow \left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = -z$

$\Rightarrow (xy + z) = \left(\sqrt{1-x^2}\sqrt{1-y^2}\right)$

Squaring both sides,

$\Rightarrow (xy + z)^2 = \left(\sqrt{1-x^2}\sqrt{1-y^2}\right)^2$

$\Rightarrow (xy)^2 + z^2 + 2xyz = (1-x^2)(1-y^2)$

$\Rightarrow x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$